

On Symmetries of the Feinberg-Zee Random Hopping Matrix

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Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. In this paper we study the spectrum Σ of the infinite Feinberg-Zee random hopping matrix, a tridiagonal matrix with zeros on the main diagonal and random ± 1 's on the first sub- and super-diagonals; the study of this non-selfadjoint random matrix was initiated in Feinberg and Zee (*Phys. Rev. E* **59** (1999), 6433–6443). Recently Hagger (*Random Matrices: Theory Appl.*, **4** 1550016 (2015)) has shown that the so-called *periodic part* Σ_π of Σ , conjectured to be the whole of Σ and known to include the unit disk, satisfies $p^{-1}(\Sigma_\pi) \subset \Sigma_\pi$ for an infinite class \mathcal{S} of monic polynomials p . In this paper we make very explicit the membership of \mathcal{S} , in particular showing that it includes $P_m(\lambda) = \lambda U_{m-1}(\lambda/2)$, for $m \geq 2$, where $U_n(x)$ is the Chebychev polynomial of the second kind of degree n . We also explore implications of these inverse polynomial mappings, for example showing that Σ_π is the closure of its interior, and contains the filled Julia sets of infinitely many $p \in \mathcal{S}$, including those of P_m , this partially answering a conjecture of the second author.

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1. Introduction

In this paper we study spectral properties of infinite matrices of the form

$$A_c = \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & c_0 & \boxed{0} & 1 \\ & & & c_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad (1.1)$$

where $c \in \Omega := \{\pm 1\}^{\mathbb{Z}}$ is an infinite sequence of ± 1 's, and the box marks the entry at $(0, 0)$. Let ℓ^2 denote the linear space of those complex-valued sequences $\phi : \mathbb{Z} \rightarrow \mathbb{C}$ for which $\|\phi\|_2 := \{\sum_{n \in \mathbb{Z}} |\phi_n|^2\}^{1/2} < \infty$, a Hilbert space equipped with the norm $\|\cdot\|_2$. Then to each matrix A_c with $c \in \Omega$ corresponds a bounded linear mapping $\ell^2 \mapsto \ell^2$, which we denote again by A_c , given by the rule

$$(A_c \phi)_m = c_m \phi_{m-1} + \phi_{m+1}, \quad m \in \mathbb{Z},$$

for $\phi \in \ell^2$.

Following [5] we will term (1.1) a *Feinberg-Zee hopping matrix*. Further, in the case where each c_m is an independent realisation of a random variable with probability measure whose support is $\{-1, 1\}$, we will term A_c a *Feinberg-Zee random hopping matrix*, this particular non-selfadjoint random matrix studied previously in [12, 13, 8, 19, 3, 5, 4, 15, 16, 17].¹ The spectrum of a realisation A_c of this random hopping matrix is given, almost surely, by (e.g., [3])

$$\text{spec } A_c = \Sigma := \bigcup_{b \in \Omega} \text{spec } A_b. \quad (1.2)$$

Here $\text{spec } A_b$ denotes the spectrum of A_b as an operator on ℓ^2 . Note that (1.2) implies that Σ is closed.

Equation (1.2) holds whenever $c \in \Omega$ is *pseudo-ergodic*, which means simply that every finite sequence of ± 1 's appears as a consecutive sequence somewhere in the infinite vector c ; it is easy to see that c is pseudo-ergodic almost surely if c is random. The concept of pseudo-ergodicity dates back to [9], as do the arguments that (1.2) holds, or see [4] for (1.2) derived as a special case of more general limit operator results.

Many of the above cited papers are concerned primarily with computing upper and lower bounds on Σ . A standard upper bound for the spectrum is provided by the numerical range. It is shown in [4] that, if $c \in \Omega$ is pseudo-ergodic, its numerical range $W(A_c)$, defined by $W(A_c) := \{(A_c \phi, \phi) : \phi \in \ell^2, \|\phi\|_2 = 1\}$, where

¹These random hopping matrices appear to have been studied initially in [12], in which paper the first superdiagonal is also a sequence of random ± 1 's. But it is no loss of generality to restrict attention to matrices of the form (1.1) as the case where the superdiagonal is also random can be reduced to (1.1) by a simple gauge transformation; see [12] or [4, Lemma 3.2, Theorem 5.1].

(\cdot, \cdot) is the inner product on ℓ^2 , is given by

$$W(A_c) = \Delta := \{x + iy : x, y \in \mathbb{R}, |x| + |y| < 2\}. \quad (1.3)$$

This gives the upper bound that $\Sigma \subset \overline{\Delta}$, the closure of Δ . Other, sharper upper bounds on Σ are discussed in Section 2 below.

This current paper is related to the problem of computing lower bounds for Σ via (1.2). If $b \in \Omega$ is constant then A_b is a Laurent matrix and $\text{spec } A_b = [-2, 2]$ if $b_m \equiv 1$, while $\text{spec } A_b = i[-2, 2]$ if $b_m \equiv -1$; thus, by (1.2), $\pi_1 := [-2, 2] \cup i[-2, 2] \subset \Sigma$. Generalising this, if $b \in \Omega$ is periodic with period n then $\text{spec } A_b$ is the union of a finite number of analytic arcs which can be computed by calculating eigenvalues of $n \times n$ matrices (see Lemma 2.2 below). And, by (1.2), $\pi_n \subset \Sigma$, where π_n is the union of $\text{spec } A_b$ over all b with period n . This implies, since Σ is closed, that

$$\Sigma_\pi := \overline{\pi_\infty} \subset \Sigma, \quad (1.4)$$

where $\pi_\infty := \bigcup_{n \in \mathbb{N}} \pi_n$.

We will call Σ_π the *periodic part* of Σ , noting that [3] conjectures that equality holds in (1.4), i.e. that π_∞ is dense in Σ and $\Sigma_\pi = \Sigma$. Whether or not this holds is an open problem, but it has been shown in [5] that π_∞ is dense in the open unit disk $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, so that

$$\overline{\mathbb{D}} \subset \Sigma_\pi \subset \Sigma. \quad (1.5)$$

For a polynomial p and $S \subset \mathbb{C}$, we define, as usual, $p(S) := \{p(\lambda) : \lambda \in S\}$ and $p^{-1}(S) := \{\lambda \in \mathbb{C} : p(\lambda) \in S\}$. (We will use throughout that if S is open then $p^{-1}(S)$ is open (p is continuous) and, if p is non-constant, then $p(S)$ is also open, e.g., [22, Theorem 10.32].) The proof of (1.5) in [5] depends on the result, in the case $p(\lambda) = \lambda^2$, that

$$p^{-1}(\pi_\infty) \subset \pi_\infty, \text{ so that also } p^{-1}(\Sigma_\pi) \subset \Sigma_\pi. \quad (1.6)$$

This implies that $S_n \subset \pi_\infty$, for $n = 0, 1, \dots$, where $S_0 := [-2, 2]$ and $S_n := p^{-1}(S_{n-1})$, for $n \in \mathbb{N}$. Thus $\bigcup_{n \in \mathbb{N}} S_n$, which is dense in $\overline{\mathbb{D}}$, is also in π_∞ , giving (1.5).

Hagger [17] makes a large generalisation of the results of [5], showing the existence of an infinite family, \mathcal{S} , containing monic polynomials of arbitrarily high degree, for which (1.6) holds. For each of these polynomials p let

$$U(p) := \bigcup_{n=1}^{\infty} p^{-n}(\mathbb{D}). \quad (1.7)$$

(Here $p^{-2}(S) := p^{-1}(p^{-1}(S))$, $p^{-3}(S) := p^{-1}(p^{-2}(S))$, etc.) Hagger [17] observes that, as a consequence of (1.5) and (1.6), $U(p) \subset \Sigma_\pi$. He also notes that standard results of complex dynamics (e.g., [11, Corollary 14.8]) imply that $J(p) \subset \overline{U(p)}$, so that $J(p) \subset \Sigma_\pi$; here $J(p)$ denotes the Julia set of the polynomial p . (Where $p^2(\lambda) := p(p(\lambda))$, $p^3(\lambda) := p(p^2(\lambda))$, etc., we recall [11] that the *filled Julia set* $K(p)$ of a polynomial p of degree ≥ 2 is the compact set of those $\lambda \in \mathbb{C}$ for which the

sequence $(p^n(\lambda))_{n \in \mathbb{N}}$, the *orbit* of λ , is bounded. Further, the boundary of $K(p)$, $J(p) := \partial K(p) \subset K(p)$, is the *Julia set* of p .)

The definition of the set \mathcal{S} in [17], while constructive, is rather indirect. The first contribution of this paper (Section 3) is to make explicit the membership of \mathcal{S} . As a consequence we show, in particular, that $P_m \in \mathcal{S}$, for $m = 2, 3, \dots$, where $P_m(\lambda) := \lambda U_{m-1}(\lambda/2)$, and U_n is the Chebychev polynomial of the second kind of degree n [1].

The second contribution of this paper (Section 4) is to say more about the interior points of Σ_π . Previous calculations of large subsets of π_∞ , precisely calculations of π_n for n as large as 30 [3, 4], suggest that Σ_π fills most of the square Δ , but $\text{int}(\Sigma_\pi)$, the interior of Σ_π , is known only to contain \mathbb{D} . Using that the whole family $\{P_m : m \geq 2\} \subset \mathcal{S}$, we prove that $(-2, 2) \subset \text{int}(\Sigma_\pi)$. This result is then used to show that Σ_π is the closure of its interior. Using that $p^{-1}(\mathbb{D}) \subset \Sigma_\pi$, for $p \in \mathcal{S}$, we also, in Section 3.2, construct new explicit subsets of Σ_π and its interior; in particular, extending (1.5), we show that $\alpha \overline{\mathbb{D}} \subset \Sigma_\pi$ for $\alpha = 1.1$.

In the final Section 5 of the paper we address a conjecture of Hagger [17] that, not only for every $p \in \mathcal{S}$ is $J(p) \subset \overline{U(p)}$ (which implies $J(p) \subset \Sigma_\pi$), but also the filled Julia set $K(p) \subset \overline{U(p)}$. This is a stronger result as, while the compact set $J(p)$ has empty interior [11, Summary 14.12], $K(p)$ contains, in addition to $J(p)$, all the bounded components of the open Fatou set $F(p) := \mathbb{C} \setminus J(p)$. We show, by a counter-example, that this conjecture is false. But, positively, we conjecture that $K(p) \subset \Sigma_\pi$ for all $p \in \mathcal{S}$, and we prove that this is true for a large subset of \mathcal{S} , in particular that $K(P_m) \subset \Sigma_\pi$ for $m \geq 2$.

The results in this paper provide new information on the almost sure spectrum $\Sigma \supset \Sigma_\pi$ of the bi-infinite Feinberg-Zee random hopping matrix. They are also relevant to the study of the spectra of the corresponding finite matrices. For $n \in \mathbb{N}$ let V_n denote the set of $n \times n$ matrices of the form (1.1), so that $V_1 := \{(0)\}$ and, for $n \geq 2$, $V_n := \{A_k^{(n)} : k = (k_1, \dots, k_n) \in \{\pm 1\}^n\}$, where

$$A_k^{(n)} := \begin{pmatrix} 0 & 1 & & & \\ k_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & k_{n-1} & 0 \end{pmatrix}. \quad (1.8)$$

(This notation will be convenient, but note that $A_k^{(n)}$ is independent of the last component of k .) Then $\text{spec } A_k^{(n)}$ is the set of eigenvalues of the matrix $A_k^{(n)}$. Let

$$\sigma_n := \bigcup_{A \in V_n} \text{spec } A, \text{ for } n \in \mathbb{N}, \text{ and } \sigma_\infty := \bigcup_{n=1}^{\infty} \sigma_n, \quad (1.9)$$

so that σ_∞ is the union of all eigenvalues of finite matrices of the form (1.8). Then, connecting spectra of finite and infinite matrices, it has been shown in [4] that $\sigma_n \subset \pi_{2n+2}$, for $n \in \mathbb{N}$, so that $\sigma_\infty \subset \pi_\infty \subset \Sigma_\pi$. Further, [16] shows that σ_∞ is dense in π_∞ , so that $\Sigma_\pi = \overline{\sigma_\infty}$. In Section 3.1 we build on and extend these results,

making a surprising connection between the eigenvalues of the finite matrices (1.8) and the spectra of the periodic operators associated to the polynomials in \mathcal{S} . The result we prove (Theorem 3.8), is key to the later arguments in Section 4.

2. Preliminaries and previous work

Notions of set convergence. We will say something below about set sequences, sequences approximating Σ and Σ_π from above and below, respectively. We will measure convergence in the standard Hausdorff metric $d(\cdot, \cdot)$ [18, Section 28] (or see [14]) on the space \mathbb{C}^C of compact subsets of \mathbb{C} . We will write, for a sequence $(S_n) \subset \mathbb{C}^C$ and $S \in \mathbb{C}^C$, that $S_n \searrow S$ if $d(S_n, S) \rightarrow 0$ as $n \rightarrow \infty$ and $S \subset S_n$ for each n ; equivalently, $S_n \searrow S$ as $n \rightarrow \infty$ if $S \subset S_n$ for each n and, for every $\epsilon > 0$, $S_n \subset S + \epsilon\mathbb{D}$, for all sufficiently large n . Similarly, we will write that $S_n \nearrow S$ if $d(S_n, S) \rightarrow 0$ as $n \rightarrow \infty$ and $S_n \subset S$ for each n ; equivalently, $S_n \nearrow S$ if $S_n \subset S$ for each n and, for every $\epsilon > 0$, $S \subset S_n + \epsilon\mathbb{D}$, for all sufficiently large n . The following observation, which follows immediately from [14, Proposition 3.6] (or see [6, Lemma 4.15]), will be useful:

Lemma 2.1. *If $S_1 \subset S_2 \subset \dots \subset \mathbb{C}$ are closed and $S_\infty := \bigcup_{n=1}^\infty S_n$ is bounded, then $S_n \nearrow \overline{S_\infty}$, as $n \rightarrow \infty$.*

Spectra of periodic operators. We will need explicit formulae for the spectra of operators A_c with $c \in \Omega$ in the case when c is periodic. For $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, let A_k^{per} denote A_c in the case that $c_{m+n} = c_n$, for $m \in \mathbb{Z}$, and $c_m = k_m$, for $m = 1, 2, \dots, n$. For $n \in \mathbb{N}$ let I_n denote the order n identity matrix, R_n the $n \times n$ matrix which is zero except for the entry 1 in row n , column 1, and let R_n^T denote the transpose of R_n . For $n \in \mathbb{N}$, $k \in \{\pm 1\}^n$, and $\varphi \in \mathbb{R}$, let $a_k(\varphi) := A_k^{(n)} + e^{-i\varphi} R_n + k_n e^{i\varphi} R_n^T$. The following characterisation of the spectra of periodic operators is well-known (see Lemma 1 and the discussion in [16]).

Lemma 2.2. *For $n \in \mathbb{N}$ and $k \in \{\pm 1\}^n$,*

$$\text{spec } A_k^{\text{per}} = \{\lambda \in \mathbb{C} : \det(a_k(\varphi) - \lambda I_n) = 0 \text{ for some } \varphi \in [0, 2\pi)\}.$$

Key to our arguments will be an explicit expansion for the determinant in the above lemma, expressed in terms of the following notation. For $n \in \mathbb{N}$, $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, and $\lambda \in \mathbb{C}$, let

$$q_k(\lambda) := \begin{vmatrix} \lambda & 1 & & & \\ k_1 & \ddots & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & k_n & \lambda \end{vmatrix}. \quad (2.1)$$

For $i, j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, let $k(i : j) := (k_i, \dots, k_j)$, for $1 \leq i \leq j \leq n$, and define

$$q_{k(i:j)}(\lambda) := \begin{cases} \lambda, & \text{if } i - j = 1, \\ 1, & \text{if } i - j = 2, \\ 0, & \text{if } i - j = 3. \end{cases} \quad (2.2)$$

Then, for $n \in \mathbb{N}$ and $k \in \{\pm 1\}^n$, expanding the determinant (2.1) by Laplace's rule by the first row and by the last row, we see that

$$q_k(\lambda) = \lambda q_{k(2:n)}(\lambda) - k_1 q_{k(3:n)}(\lambda) = \lambda q_{k(1:n-1)}(\lambda) - k_n q_{k(1:n-2)}(\lambda). \quad (2.3)$$

The following lemma follows easily by induction on n , using (2.3). The bounds on q_k stated are used later in Corollary 5.2.

Lemma 2.3. *If $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$, then q_k is a monic polynomial of degree $n+1$, and q_k is even and $q_k(0) = \pm 1$ if n is odd, q_k is odd and $q_k(0) = 0$ if n is even. Further, $|q_k(\lambda)| \geq |q_{k(1:n-1)}(\lambda)| + 1$, $|q_k(\lambda)| \geq |q_{k(2:n)}(\lambda)| + 1$, and $|q_k(\lambda)| \geq n + 2$, for $|\lambda| \geq 2$.*

For $n \in \mathbb{N}$ let J_n denote the $n \times n$ flip matrix, that is the $n \times n$ matrix with entry $\delta_{i, n+1-j}$ in row i , column j , where $\delta_{i,j}$ is the Kronecker delta. Then $J_n^2 = I_n$ so that $(\det J_n)^2 = 1$. For $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, let $k' := kJ_n = (k_n, \dots, k_1)$. The first part of the following lemma is essentially a particular instance of a general property of determinants.

Lemma 2.4. *If $k \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$, and $\ell = k'$, then $q_k = q_\ell$; if $\ell = -k$, then*

$$q_\ell(\lambda) = i^{-n-1} q_k(i\lambda). \quad (2.4)$$

Proof. Suppose first that $\ell = k'$. Then $q_k(\lambda)$ given by (2.3) is the determinant of a matrix A , and $q_\ell(\lambda)$ is the determinant of $J_{n+1} A^T J_{n+1}$, so that $q_\ell(\lambda) = (\det J_{n+1})^2 \det A = q_k(\lambda)$. That (2.4) holds if $\ell = -k$ can be shown by an easy induction on n , using (2.3), or directly by a gauge transformation. \square

Here is the announced explicit expression for the determinant in Lemma 2.2.

Lemma 2.5. [17, Lemma 3] *For $n \in \mathbb{N}$, $k \in \{\pm 1\}^n$, $\lambda \in \mathbb{C}$, and $\varphi \in [0, 2\pi)$,*

$$(-1)^n \det(a_k(\varphi) - \lambda I_n) = p_k(\lambda) - e^{i\varphi} \prod_{j=1}^n k_j - e^{-i\varphi},$$

where p_k is a monic polynomial of degree n given by

$$(-1)^n p_k(\lambda) = q_{k(1:n-1)}(-\lambda) - k_n q_{k(2:n-2)}(-\lambda). \quad (2.5)$$

Further, p_k is odd (even) if n is odd (even).

Since, from the above lemmas, p_k is odd (even) and q_k even (odd) if n is odd (even), (2.5) implies that

$$p_k(\lambda) = q_{k(1:n-1)}(\lambda) - k_n q_{k(2:n-2)}(\lambda) \quad (2.6)$$

$$= q_{k(1:n-1)}(\lambda) + q_{k(2:n)}(\lambda) - \lambda q_{k(2:n-1)}(\lambda), \quad (2.7)$$

this last equation obtained using (2.3). The following lemma, proved using these representations, makes clear that many different vectors k correspond to the same polynomial p_k .

Lemma 2.6. *If $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$, and $\ell = k'$ or ℓ is a cyclic permutation of k , then $p_k = p_\ell$. If $\ell = -k$ then $p_\ell(\lambda) = i^{-n} p_k(i\lambda)$.*

Proof. Using (2.6) and (2.3) we see that

$$\begin{aligned} p_\ell(\lambda) - p_k(\lambda) &= q_{\ell(1:n-1)}(\lambda) - q_{k(1:n-1)}(\lambda) + k_n q_{k(2:n-2)}(\lambda) - \ell_n q_{\ell(2:n-2)}(\lambda) \\ &= \lambda (q_{\ell(2:n-1)}(\lambda) - q_{k(1:n-2)}(\lambda)) - \ell_1 q_{\ell(3:n-1)}(\lambda) \\ &\quad + k_{n-1} q_{k(1:n-3)}(\lambda) + k_n q_{k(2:n-2)}(\lambda) - \ell_n q_{\ell(2:n-2)}(\lambda). \end{aligned}$$

If ℓ is a cyclic shift of k , i.e., $\ell_j = k_{j-1}$, $j = 2, \dots, n$, and $\ell_1 = k_n$, then the right-hand side is identically zero. Thus $p_\ell = p_k$ if ℓ is a cyclic permutation of k .

If $\ell = k'$ then that $p_k = p_\ell$ follows from (2.7) and Lemma 2.4. If $\ell = -k$ then that $p_\ell(\lambda) = i^{-n} p_k(i\lambda)$ follows from (2.6) and Lemma 2.4. \square

Call $k \in \{\pm 1\}^n$ *even* if $\prod_{j=1}^n k_j = 1$, and *odd* if $\prod_{j=1}^n k_j = -1$. Then [17, Corollary 5], it is immediate from Lemmas 2.2 and 2.5 that

$$\text{spec } A_k^{\text{per}} = p_k^{-1}([-2, 2]), \text{ if } k \text{ is even, } \text{spec } A_k^{\text{per}} = p_k^{-1}(i[-2, 2]), \text{ if } k \text{ is odd.} \quad (2.8)$$

Complex dynamics. In Section 5 below we show that filled Julia sets, $K(p)$, of particular polynomials p , are contained in the periodic part Σ_π of the almost sure spectrum of the Feinberg-Zee random hopping matrix. To articulate and prove these results we will need terminology and results from complex dynamics.

Throughout this section p denotes a polynomial of degree ≥ 2 . We have defined above the compact set that is the filled Julia set $K(p)$, the Julia set $J(p) = \partial K(p) \subset K(p)$, the Fatou set $F(p)$ (the open set that is the complement of $J(p)$), and the orbit of $z \in \mathbb{C}$. It is easy to see that, if $z \notin K(p)$, i.e., the orbit of z is not bounded, then $p^n(z) \rightarrow \infty$ as $n \rightarrow \infty$, i.e., $z \in A_p(\infty)$, the *basin of attraction of infinity*. We call $S \subset \mathbb{C}$ *invariant* if $p(S) = S$, and *completely invariant* if both S and its complement are invariant, which holds iff $p^{-1}(S) = S$. Clearly $A_p(\infty)$ is completely invariant.

We call z a *fixed point* of p if $p(z) = z$ and a *periodic point* if $p^n(z) = z$, for some $n \in \mathbb{N}$, in which case the finite sequence $(z_0, z_1, \dots, z_{n-1})$, where $z_0 = z$, $z_1 = p(z_0)$, ..., is the *cycle* of the periodic point z . We say that z is an *attracting fixed point* if $|p'(z)| < 1$, a *repelling fixed point* if $|p'(z)| > 1$, and a *neutral fixed point* if $|p'(z)| = 1$. Generalising, we say that a periodic point z is attracting/repelling/neutral if $|P'(z)| < 1/> 1/= 1$, where $P = p^n$. By the chain rule, $P'(z) = p'(z_0) \dots p'(z_{n-1})$, where $z_0 = z$ and $z_j := p(z_{j-1})$, $j = 1, \dots, n-1$.

The value $\gamma = P'(z)$ is the *multiplier* of the neutral periodic point z (clearly $|\gamma| = 1$). If z is a neutral periodic point with multiplier γ we say that it is *rationally neutral* if $\gamma^N = 1$ for some $N \in \mathbb{N}$, otherwise we say that it is *irrationally neutral*. We call z a *critical point* of p if $p'(z) = 0$.

If w is an attracting periodic point we denote by $A_p(w)$ the *basin of attraction* of the cycle $C = \{z_0, \dots, z_{n-1}\}$ of z , by which we mean $A_p(w) := \{z \in \mathbb{C} : d(p^n(z), C) \rightarrow 0 \text{ as } n \rightarrow \infty\}$. Here, for $S \subset \mathbb{C}$ and $z \in \mathbb{C}$, $d(z, S) := \inf_{w \in S} |z - w|$. It is easy to see that $A_p(w)$ contains some neighbourhood of C , and hence that $A_p(w)$ is open.

We will make use of standard properties of the Julia set captured in the following theorem. We recall that a family F of analytic functions is *normal* at a point $z \in \mathbb{C}$ if, in some fixed neighbourhood N of z , each $f \in F$ is analytic and every sequence drawn from F has a subsequence that is convergent uniformly either to some analytic function or to ∞ .

Theorem 2.7. [11, Summary 14.12] *$J(p)$ is compact with no isolated points, is uncountable, and has empty interior. $J(p)$ is completely invariant, $J(p) = J(p^n)$, for every $n \in \mathbb{N}$,*

$$J(p) = \{z \in \mathbb{C} : \text{the family } \{p^1, p^2, \dots\} \text{ is not normal at } z\}, \quad (2.9)$$

$J(p)$ is the closure of the repelling periodic points of p , and, for all except at most one $z \in \mathbb{C}$,

$$J(p) \subset \overline{\bigcup_{n=1}^{\infty} p^{-n}(\{z\})}. \quad (2.10)$$

The Fatou set $F(p)$ has one unbounded component U . It follows from (2.9) that $U \subset A_p(\infty)$; indeed, $U = A_p(\infty)$ as a consequence of the maximum principle [2]. It may happen that this is the only component of $F(p)$ so that $A_p(\infty) = F(p)$. This is the case if $k = (1, 1)$ and $p(z) = p_k(z) = z^2 - 2$, when $K(p) = J(p) = [-2, 2]$ [2, p. 55]. If $F(p)$ has more than one component it either has two components (for example, if $k = (-1, 1)$ and $p(z) = p_k(z) = z^2$, when $K(p) = \overline{\mathbb{D}}$ and $J(p) = \partial\mathbb{D}$), or infinitely many components [2, Theorem IV.1.2]. It follows from (2.10) that $J(p) = \partial A_p(\infty)$ and $J(p) = \partial A_p(w)$ if w is an attracting fixed point or periodic point [2, Theorem III.2.1]. Arguing similarly [2, Theorem 1.7], $J(p) = \partial F_B(p)$, where $F_B(p) := \text{int}(K(p)) = F(p) \setminus A_p(\infty)$, so that $K(p) = \overline{F_B(p)}$.

Because $J(p)$ is completely invariant and p is an open map, the image $V = p(U)$ of any component U of $F(p)$ is also a component of $F(p)$. Now consider the orbit of U , i.e. $(p^n(U))_{n=1}^{\infty}$. The following statement of possible behaviours is essentially Sullivan's theorem [2, pp. 69-71].

Theorem 2.8. *Let U be a component of $F(p)$. Then one of the following cases holds:*

i) $p^n(U) = U$, for some $n \in \mathbb{N}$, in which case we call U a periodic component of $F(p)$, and call the smallest n for which $p^n(U) = U$ the period of U . (If $n = 1$, when U is invariant, we also term U a fixed component of $F(p)$.)

ii) $p^r(U)$ is a periodic component of $F(p)$ for some $r \in \mathbb{N}$, in which case we say that U is a preperiodic component of $F(p)$.

The above theorem makes clear that the orbit of every component U of $F(p)$ enters a periodic cycle after a finite number of steps. To understand the eventual

fate under iterations of p of the components of the Fatou set it is helpful to understand the possible behaviours of a periodic component. This is achieved in the *classification theorem* (e.g. [2]). To state this theorem we introduce further terminology. Let us call a fixed component U of $F(p)$ a *parabolic* component if there exists a neutral fixed point $w \in \partial U$ with multiplier 1 such that the orbit of every $z \in U$ converges to w . Call a fixed component U of $F(p)$ a *Siegel disk* if p is conjugate to an irrational rotation on U , which means that there exists a conformal mapping $\varphi : U \rightarrow V$ with $0 \in V$ and an irrational $\theta \in \mathbb{R}$ such that

$$\varphi(p(z)) = g(\varphi(z)) = \gamma\varphi(z), \quad z \in U, \quad (2.11)$$

where $g(w) = \gamma w$, $w \in V$, and $\gamma = \exp(2\pi i\theta)$. It is easy to see that, for $w \in U$, $p(w) = w$ iff $w = \varphi^{-1}(0)$, and $p'(w) = \gamma$. Thus every Siegel disk contains a unique irrationally neutral fixed point (the Siegel disk fixed point).

Theorem 2.9. Classification Theorem [2, Theorem IV.2.1]. *If U is a periodic component of $F(p)$ with period $n \in \mathbb{N}$, in which case U is also a component of $F(p^n) = F(p)$, then exactly one of the following holds:²*

- a) U contains an attracting periodic point w which is an attracting fixed point of p^n , and $U \subset A_p(w)$;
- b) U is a parabolic component of $F(p^n)$;
- c) U is a Siegel disk component of $F(p^n)$.

The following proposition relates the above cases to critical points of p (see [2, Theorems III 2.2 and 2.3, pp. 83-84]):

Proposition 2.10. *If U is a periodic component of $F(p)$ with period n then either: (i) U is a parabolic component of $F(p^n)$ or contains an attracting periodic point, in which case $\cup_{m=1}^n p^m(U)$ contains a critical point of p ; or (ii) U is a Siegel disk component of $F(p^n)$ and there is a critical point w of p such that the orbit of w is dense in ∂U .*

The following proposition will do the work for us in Section 5.

Proposition 2.11. *Suppose that $S \subset \mathbb{C}$ is bounded, open and simply-connected, that $T \subset S$ is closed, and that the orbit of every critical point in $K(p)$ eventually lies in T . Then*

$$K(p) \subset G := \overline{\bigcup_{n=1}^{\infty} p^{-n}(S)}.$$

Proof. That $z \in G$ if $z \in J(p)$ follows from (2.10). If $z \in F_B(p)$ then, by Theorems 2.8 and 2.9, after a finite number of iterations the orbit of z is in a periodic component of $F(p)$ that is parabolic, part of the domain of attraction of an attracting periodic point, or is a Siegel disk. In the first two cases it follows that $d(p^n(z), C) \rightarrow 0$ as $n \rightarrow \infty$ for some cycle C , but also $d(p^n(w), C) \rightarrow 0$ for some critical point w by Proposition 2.10. This last implies that $C \cap T$ is non-empty,

²The result as stated for rational p in [2] gives a 4th option, that U is a *Herman ring component* of $F(p^n)$. This option is excluded if p is a polynomial [21, p. 166].

and so $p^n(z) \in S$ for some n . In the case that the orbit of z is eventually in a Siegel disk then also $p^n(z) \in S$ for some n for, if the orbit of every critical point $w \in K(p)$ is eventually in T , it follows that the boundary of every Siegel disk is in T , and (as S is simply connected) that every Siegel disk is in S . \square

Previous upper bounds on Σ . We have noted above that, if $c \in \Omega$ is pseudo-ergodic, then $\Sigma = \text{spec } A_c \subset \overline{W(A_c)} = \overline{\Delta}$, given by (1.3). Similarly, the spectrum of A_c^2 is contained in the closure of its numerical range, so that³

$$\Sigma \subset \{\pm\sqrt{z} : z \in \text{spec}(A_c^2)\} \subset N_2 := \{\pm\sqrt{z} : z \in \overline{W(A_c^2)}\}. \quad (2.12)$$

Hagger [15] introduces a new, general method for computing numerical ranges of infinite tridiagonal matrices via the Schur test, which he applies to computing the numerical range of A_c^2 (by expressing it as the direct sum of tridiagonal matrices). These calculations show that $\Sigma \subset N_2 \subsetneq \overline{\Delta}$; indeed, the calculations in [15] imply that $N_2 = \{r \exp(i\theta) : 0 \leq r \leq \rho(\theta), 0 \leq \theta < 2\pi\}$, where $\rho \in C(\mathbb{R})$ is even and periodic with period $\pi/2$, given explicitly on $[-\pi/4, \pi/4]$ by

$$\rho(\theta) = \begin{cases} \sqrt{2}, & \pi/6 \leq |\theta| \leq \pi/4, \\ 2/\sqrt{\cos 2\theta + \sqrt{3} |\sin 2\theta|}, & |\theta| \leq \pi/6. \end{cases} \quad (2.13)$$

By comparison, in polar form,

$$W(A_c) = \Delta = \{re^{i\theta} : 0 \leq r < 2/(|\cos \theta| + |\sin \theta|), 0 \leq \theta < 2\pi\}. \quad (2.14)$$

Figure 2 includes a visualisation of $\overline{\Delta}$ and N_2 .

The bound (2.12), expressed concretely through (2.13), is the sharpest explicit upper bound on Σ obtained to date. It implies that Σ is not convex since we also know (see (3.1) below) that ± 2 , $\pm 2i$, and $\pm 1 \pm i$ are all in Σ .

A different family of upper bounds was established in [4] (and see [7]), expressed in terms of pseudospectra. For a square matrix A of order n and $\epsilon > 0$ let $\text{spec}_\epsilon A$ denote the ϵ -pseudospectrum of A (with respect to the 2-norm), i.e., $\text{spec}_\epsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I_n)^{-1}\|_2 > \epsilon^{-1}\}$, with the understanding that $\|(A - \lambda I)^{-1}\|_2 = \infty$ if λ is an eigenvalue, so that $\text{spec } A \subset \text{spec}_\epsilon A$. (Here $\|\cdot\|_2$ is the operator norm of a linear mapping on \mathbb{C}^n equipped with the 2-norm.) Analogously to (1.9), for $\epsilon > 0$ and $n \in \mathbb{N}$, let

$$\sigma_{n,\epsilon} := \bigcup_{A \in V_n} \text{spec}_\epsilon A, \quad (2.15)$$

which is the union of the pseudospectra of 2^{n-1} distinct matrices. Then it is shown in [4] that

$$\Sigma_n^* := \overline{\sigma_{n,\epsilon_n}} \searrow \Sigma \text{ as } n \rightarrow \infty, \quad (2.16)$$

where $\epsilon_n := 4 \sin \theta_n \leq 2\pi/(n+2)$ and θ_n is the unique solution in $(\pi/(2n+6), \pi/(2n+4)]$ of the equation $2 \cos((n+1)\theta) = \cos((n-1)\theta)$.

³Equation (2.12) is the idea behind higher order numerical ranges; indeed, where p is the polynomial $p(\lambda) = \lambda^2$, N_2 is $\text{Num}(p, A_c)$ in the notation of [10, p. 278], so that N_2 is a superset of the second order numerical range.

Clearly, $\{\Sigma_n^* : n \in \mathbb{N}\}$ is a convergent family of upper bounds for Σ that is in principle computable; deciding whether $\lambda \in \Sigma_n^*$ requires only computation of smallest singular values of $n \times n$ matrices (see [4, (39)]). Explicitly $\Sigma_1^* = 2\overline{\mathbb{D}}$, and Σ_n^* is plotted for $n = 6, 12, 18$ in [4]. But for these values $\Sigma_n^* \supset \Delta$, and computing Σ_n^* for larger n is challenging, requiring computation of the smallest singular value of 2^{n-1} matrices of order n to decide whether a particular $\lambda \in \Sigma_n^*$. Substantial numerical calculations in [4] established that $1.5 + 0.5i \notin \Sigma_{34}^*$, providing the first proof that Σ is a strict subset of $\overline{\Delta}$, this confirmed now by the simple explicit bound (2.12) and (2.13).

3. Lower Bounds on Σ and Symmetries of Σ and Σ_π

Complementing the upper bounds on Σ that we have just discussed, lower bounds on Σ have been obtained by two methods of argument. The first is that (1.2) tells us that $\text{spec } A_b \subset \Sigma$ for every $b \in \Omega$. In particular this holds in the case when b is periodic, when the spectrum of A_b is given explicitly by Lemmas 2.2 and 2.5, so that, as observed in the introduction,

$$\pi_n := \bigcup_{k \in \{\pm 1\}^n} \text{spec } A_k^{\text{per}} \subset \Sigma.$$

Explicitly [4, Lemma 2.6], in particular,

$$\pi_1 = [-2, 2] \cup i[-2, 2] \quad \text{and} \quad \pi_2 = \pi_1 \cup \{x \pm ix : -1 \leq x \leq 1\}. \quad (3.1)$$

In the introduction we have defined $\pi_\infty := \bigcup_{n=1}^\infty \pi_n$ and have termed $\Sigma_\pi := \overline{\pi_\infty}$, also a subset of Σ since Σ is closed, the periodic part of Σ . We have also recalled the conjecture of [3] that $\Sigma_\pi = \Sigma$. Let

$$\Pi_n := \bigcup_{m=1}^n \pi_m \subset \pi_\infty \subset \Sigma_\pi \subset \Sigma.$$

Then it follows from Lemma 2.1 that

$$\Pi_n \nearrow \Sigma_\pi \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

If, as conjectured, $\Sigma_\pi = \Sigma$, then (3.2) complements (2.16); together they sandwich Σ by convergent sequences of upper (Σ_n^*) and lower (Π_n) bounds that can both be computed by calculating eigenvalues of $n \times n$ matrices. Figures 2 and 3 include visualisations of π_{30} , indistinguishable by eye from Π_{30} , but note that the solid appearance of π_{30} , which is the union of a large but finite number of analytic arcs, is illusory. See [3, 4] for visualisations of π_n for a range of n , suggestive that the convergence (3.2) is approximately achieved by $n = 30$.

The same method of argument (1.2) to obtain lower bounds was used in [3], where a special sequence $b \in \Omega$ was constructed with the property that $\text{spec } A_b \supset \overline{\mathbb{D}}$, so that, by (1.2), $\overline{\mathbb{D}} \subset \Sigma$. The stronger result (1.5), that this new lower bound on Σ is in fact also a subset of Σ_π , was shown in [5], via a second method of argument for constructing lower bounds, based on surprising symmetries of Σ and

Σ_π . We will spell out in a moment these symmetries (one of these described first in [5], the whole infinite family in [17]), which will be both a main subject of study and a main tool for argument in this paper. But first we note more straightforward but important symmetries. In this lemma and throughout $\bar{\lambda}$ denotes the complex conjugate of $\lambda \in \mathbb{C}$.

Lemma 3.1. [4, Lemma 3.4] (and see [19], [5, Lemma 4]). *All of π_n , σ_n , Σ_π , and Σ are invariant with respect to the maps $\lambda \mapsto i\lambda$ and $\lambda \mapsto \bar{\lambda}$, and so are invariant under the dihedral symmetry group D_2 generated by these two maps.*

To expand on the brief discussion in the introduction, [17] proves the existence of an infinite set \mathcal{S} of monic polynomials of degree ≥ 2 , this set defined constructively in the following theorem, such that the elements $p \in \mathcal{S}$ are symmetries of π_∞ and Σ in the sense that (3.3) below holds.

Theorem 3.2. [17] *Let \mathcal{S} denote the set of those polynomials p_k , defined by (2.5), with $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$ for some $n \geq 2$, for which it holds that: (i) $k_{n-1} = -1$ and $k_n = 1$; (ii) $p_k = p_{\hat{k}}$, where $\hat{k} \in \{\pm 1\}^n$ is the vector identical to k but with the last two entries interchanged, so that $\hat{k}_{n-1} = 1$ and $\hat{k}_n = -1$. Then*

$$\Sigma \subset p(\Sigma) \text{ and } p^{-1}(\pi_\infty) \subset \pi_\infty, \quad (3.3)$$

for all $p \in \mathcal{S}$.

We will call \mathcal{S} Hagger's set of polynomial symmetries for Σ .

We remark that if $p \in \mathcal{S}$ then it follows from (3.3), by taking closures and recalling that p is continuous, that also

$$p^{-1}(\Sigma_\pi) \subset \Sigma_\pi \text{ and } p^{-1}(\text{int}(\Sigma_\pi)) \subset \text{int}(\Sigma_\pi). \quad (3.4)$$

We note also that $p^{-1}(\pi_\infty) \subset \pi_\infty$ implies that $\pi_\infty \subset p(\pi_\infty)$, but not vice versa, and that $\Sigma \subset p(\Sigma)$ iff

$$p^{-1}(\{\lambda\}) \cap \Sigma \neq \emptyset, \text{ for all } \lambda \in \Sigma.$$

Further, we note that it was shown earlier in [5] that (3.3) holds for the particular case $p(\lambda) = \lambda^2$ (this the only element of \mathcal{S} of degree 2, see Table 1); in [5] it was also shown, as an immediate consequence of (3.3) and Lemma 3.1, that

$$p^{-1}(\Sigma) \subset \Sigma,$$

for $p(\lambda) = \lambda^2$. Whether this last inclusion holds in fact for all $p \in \mathcal{S}$ is an open problem.

Our first result is a much more explicit characterisation of \mathcal{S} .

Proposition 3.3. *The set \mathcal{S} is given by $\mathcal{S} = \{p_k : k \in \mathcal{K}\}$, where \mathcal{K} consists of those vectors $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$ with $n \geq 2$, for which: (i) $k_{n-1} = -1$ and $k_n = 1$; and (ii) $n = 2$, or $n \geq 3$ and $k_j = k_{n-j-1}$, for $1 \leq j \leq n-2$, so that (k_1, \dots, k_{n-2}) is a palindrome. Moreover, if $k \in \mathcal{K}$, then*

$$p_k(\lambda) = \lambda q_{k(1:n-2)}(\lambda). \quad (3.5)$$

Proof. It is clear from Theorem 3.2 that what we have to prove is that, if $k \in \{\pm 1\}^n$ with $n \geq 2$ and $k_{n-1} = -1$, $k_n = 1$, then $p_k = p_{\widehat{k}}$ if $n = 2$ or 3 ; further, if $n \geq 4$, then $p_k = p_{\widehat{k}}$ iff (k_1, \dots, k_{n-2}) is a palindrome.

If $k \in \{\pm 1\}^n$ with $n \geq 2$ and $k_{n-1} = -1$, $k_n = 1$, then, from (2.6) and (2.3),

$$\begin{aligned} p_k(\lambda) &= q_{k(1:n-1)}(\lambda) - k_n q_{k(2:n-2)}(\lambda) \\ &= \lambda q_{k(1:n-2)}(\lambda) + q_{k(1:n-3)}(\lambda) - q_{k(2:n-2)}(\lambda). \end{aligned}$$

Thus, if $n = 2$ or 3 , or $n \geq 4$ and (k_1, \dots, k_{n-2}) is a palindrome, $p_k(\lambda) = \lambda q_{k(1:n-2)}(\lambda)$ since $q_{k(1:n-3)}(\lambda) = q_{k(2:n-2)}(\lambda)$, this a consequence of the definitions (2.2) in the cases $n = 2$ and 3 , of Lemma 2.4 in the case $n \geq 4$. Similarly, $p_{\widehat{k}}(\lambda) = \lambda q_{k(1:n-2)}(\lambda)$, so that $p_k = p_{\widehat{k}}$.

Conversely, assume that $k \in \{\pm 1\}^n$ with $n \geq 4$, $k_{n-1} = -1$, $k_n = 1$, and $p_k = p_{\widehat{k}}$. To show that (k_1, \dots, k_{n-2}) is a palindrome we need to show that $k_j = k_{n-j-1}$, for $1 \leq j \leq (n-2)/2$. Using (2.6) and then (2.3), we see that

$$\begin{aligned} 0 = p_k(\lambda) - p_{\widehat{k}}(\lambda) &= q_{k(1:n-1)}(\lambda) - q_{\widehat{k}(1:n-1)}(\lambda) - 2q_{k(2:n-2)}(\lambda) \\ &= 2q_{k(1:n-3)}(\lambda) - 2q_{k(2:n-2)}(\lambda). \end{aligned}$$

Thus $q_{k(1:n-3)} = q_{k(2:n-2)}$. But, if $q_{k(j:n-j-2)} = q_{k(j+1:n-j-1)}$ and $1 \leq j \leq (n-2)/2$, then, applying (2.3),

$$\begin{aligned} 0 &= q_{k(j:n-j-2)}(\lambda) - q_{k(j+1:n-j-1)}(\lambda) \\ &= \lambda q_{k(j+1:n-j-2)}(\lambda) - k_j q_{k(j+2:n-j-2)}(\lambda) \\ &\quad - (\lambda q_{k(j+1:n-j-2)}(\lambda) - k_{n-j-1} q_{k(j+1:n-j-3)}(\lambda)) \\ &= -k_j q_{k(j+2:n-j-2)}(\lambda) + k_{n-j-1} q_{k(j+1:n-j-3)}(\lambda). \end{aligned}$$

As this holds for all λ and, by Lemma 2.3 and (2.2), $q_{k(j+2:n-j-2)}$ and $q_{k(j+1:n-j-3)}$ are both monic polynomials of degree $n-2j-2$, it follows first that $k_j = k_{n-j-1}$ and then that $q_{k(j+1:n-j-3)} = q_{k(j+2:n-j-2)}$. Thus that $k_j = k_{n-j-1}$ for $1 \leq j \leq (n-2)/2$ follows by induction on j . \square

The following corollary is immediate from (3.5) and Lemma 2.3.

Corollary 3.4. *Suppose that $n \geq 2$, $k \in \{\pm 1\}^n$, and $p_k \in \mathcal{S}$. Then, as $\lambda \rightarrow 0$, $p_k(\lambda) = \pm \lambda + O(\lambda^3)$ if n is odd, while $p_k(\lambda) = O(\lambda^2)$ if n is even.*

Let us denote by P_m the polynomial p_k when k has length $m \geq 2$, $k_{m-1} = -1$, $k_m = 1$, and all other entries are 1's. It is convenient also to define $P_1(\lambda) = \lambda$. Clearly, as a consequence of the above proposition, $P_m \in \mathcal{S}$ for $m \geq 2$ (that these particular polynomials are in \mathcal{S} was observed earlier in [17]). We will write down shortly an explicit formula for P_m in terms of Chebyshev polynomials of the 2nd kind. Recall that $U_n(x)$, the Chebyshev polynomial of the 2nd kind of degree n , is defined by [1] $U_0(x) := 1$, $U_1(x) := 2x$, and $U_{n+1}(x) := 2xU_n(x) - U_{n-1}(x)$, for $n \in \mathbb{N}$.

Lemma 3.5. *For $m \in \mathbb{N}$, $P_m(\lambda) = \lambda U_{m-1}(\lambda/2)$.*

Proof. This follows easily by induction from (3.5) and (2.3). \square

We note that, using the standard trigonometric representations for the Chebyshev polynomials [1], for $m \in \mathbb{N}$,

$$P_m(2 \cos \theta) = 2 \cos \theta U_{m-1}(\cos \theta) = 2 \cot \theta \sin m\theta =: r_m(\theta). \quad (3.6)$$

A similar representation in terms of hyperbolic functions can be given for the polynomial p_k when k has length $2m-1$ and $k_j = (-1)^j$; we denote this polynomial by Q_m . Clearly, for $m \geq 2$, $Q_m \in \mathcal{S}$ by Proposition 3.3, and Q_m is an odd function by Lemma 2.5. The proof of the following lemma, like that of Lemma 3.5, is a straightforward induction that we leave to the reader.

Lemma 3.6. $Q_1(\lambda) = \lambda$, $Q_2(\lambda) = \lambda^3 + \lambda$, and $Q_{m+1}(\lambda) = \lambda^2 Q_m(\lambda) + Q_{m-1}(\lambda)$, for $m \geq 2$. Moreover, for $m \in \mathbb{N}$ and $\theta \geq 0$,

$$Q_m(\sqrt{2 \sinh \theta}) = \begin{cases} \sqrt{2 \sinh \theta} \frac{\sinh(m\theta) + \cosh((m-1)\theta)}{\cosh \theta}, & \text{if } m \text{ is even,} \\ \sqrt{2 \sinh \theta} \frac{\cosh(m\theta) + \sinh((m-1)\theta)}{\cosh \theta}, & \text{if } m \text{ is odd.} \end{cases}$$

The following lemma leads, using Lemmas 3.5 and 3.6, to explicit formulae for other polynomials in \mathcal{S} . For example, if P_m^* denotes the polynomial p_k when k has length $m \geq 2$, $k_{m-1} = -1$, $k_m = 1$, and all other entries are -1 's, then, by Lemmas 3.5 and 3.7,

$$P_m^*(\lambda) = i^{-m} P_m(i\lambda) = i^{1-m} \lambda U_{m-1}(i\lambda/2). \quad (3.7)$$

Lemma 3.7. If $k \in \{\pm 1\}^n$ and $p_k \in \mathcal{S}$, then $p_{-k} \in \mathcal{S}$ and $p_{-k}(\lambda) = i^{-n} p_k(i\lambda)$.

Proof. Suppose that $k \in \{\pm 1\}^n$ and $p_k \in \mathcal{S}$. If $n = 2$, then $\widehat{k} = -k$ and $p_{-k} = p_{\widehat{k}} = p_k \in \mathcal{S}$. If $n \geq 3$, defining $\ell \in \{\pm 1\}^n$ by $\ell_{n-1} = -1$, $\ell_n = 1$, and $\ell_j = -k_j$, for $j = 1, \dots, n-2$, $p_\ell \in \mathcal{S}$ by Proposition 3.3, so that $p_{-k} = p_{\widehat{\ell}} = p_\ell \in \mathcal{S}$. That $p_{-k}(\lambda) = i^{-n} p_k(i\lambda)$ comes from Lemma 2.6. \square

We note that Proposition 3.3 implies that there are precisely $2^{\lceil \frac{n}{2} \rceil - 1}$ vectors of length n in \mathcal{K} , so that there are between 1 and $2^{\lceil \frac{n}{2} \rceil - 1}$ polynomials of degree n in \mathcal{S} , as conjectured in [17]. Note, however, that there may be more than one $k \in \mathcal{K}$ that induce the same polynomial $p_k \in \mathcal{S}$. For example, $p_k(\lambda) = \lambda^6 - \lambda^2$ for $k = (-1, 1, 1, -1, -1, 1)$, and, defining $\ell = (1, -1, -1, 1, -1, 1)$ and using Lemma 3.7, also $p_\ell(\lambda) = p_{\widehat{\ell}}(\lambda) = p_{-k}(\lambda) = -p_k(i\lambda) = \lambda^6 - \lambda^2$. In Table 1 (cf. [17]) we tabulate all the polynomials in \mathcal{S} of degree ≤ 6 .

If $p, q \in \mathcal{S}$, so that p and q are polynomial symmetries of Σ in the sense that (3.3) holds, then also the composition $p \circ q$ is a polynomial symmetry of Σ in the same sense. But note from Table 1 that, while $P_3 \circ P_2 \in \mathcal{S}$, none of $P_2 \circ P_2$, $P_2 \circ P_2 \circ P_2$, $Q_2 \circ P_2$, $P_2 \circ P_3$, or $P_2 \circ Q_2$ are in \mathcal{S} . Thus \mathcal{S} does not contain all polynomial symmetries of Σ , but whether there are polynomial symmetries that are not either in \mathcal{S} or else compositions of elements of \mathcal{S} is an open question.

We finish this section by showing in subsection 3.1 the surprising result that \mathcal{S} is large enough that we can reconstruct the whole of Σ_π from the polynomials $p_k \in \mathcal{S}$. This result will in turn be key to the proof of our main theorem in Section

k	$p_k(\lambda)$
$(-1, 1)$	$\lambda^2 = P_2(\lambda)$
$(1, -1, 1)$	$\lambda^3 - \lambda = P_3(\lambda)$
$(-1, -1, 1)$	$\lambda^3 + \lambda = Q_2(\lambda) = P_3^*(\lambda)$
$(1, 1, -1, 1)$	$\lambda^4 - 2\lambda^2 = P_4(\lambda)$
$(-1, -1, -1, 1)$	$\lambda^4 + 2\lambda^2 = P_4^*(\lambda)$
$(1, 1, 1, -1, 1)$	$\lambda^5 - 3\lambda^3 + \lambda = P_5(\lambda)$
$(1, -1, 1, -1, 1)$	$\lambda^5 - \lambda^3 + \lambda = -iQ_3(i\lambda)$
$(-1, 1, -1, -1, 1)$	$\lambda^5 + \lambda^3 + \lambda = Q_3(\lambda)$
$(-1, -1, -1, -1, 1)$	$\lambda^5 + 3\lambda^3 + \lambda = P_5^*(\lambda)$
$(1, 1, 1, 1, -1, 1)$	$\lambda^6 - 4\lambda^4 + 3\lambda^2 = P_6(\lambda)$
$(1, -1, -1, 1, -1, 1)$	$\lambda^6 - \lambda^2 = P_3(P_2(\lambda))$
$(-1, -1, -1, -1, -1, 1)$	$\lambda^6 + 4\lambda^4 + 3\lambda^2 = P_6^*(\lambda)$

TABLE 1. The elements $p_k \in \mathcal{S}$ of degree ≤ 6 .

4. Then in subsection 3.2 we use that (3.3) holds for the polynomials in Table 1 to obtain new explicit lower bounds on Σ_π , including that $1.1\overline{\mathbb{D}} \subset \Sigma_\pi \subset \Sigma$.

3.1. Connecting eigenvalues of finite matrices and polynomial symmetries of Σ

Recall from Proposition 3.3 that $\mathcal{S} = \{p_k : k \in \mathcal{K}\}$, let $K := \cup_{n \in \mathbb{N}} \{\pm 1\}^n$, and define

$$\pi_\infty^{\mathcal{S}} := \bigcup_{k \in \mathcal{K}} \text{spec } A_k^{\text{per}} \subset \pi_\infty = \bigcup_{k \in K} \text{spec } A_k^{\text{per}}. \quad (3.8)$$

The following result seems rather surprising, given that \mathcal{K} is much smaller than K in the sense that there are precisely $2^{\lceil \frac{n}{2} \rceil - 1}$ vectors of length n in \mathcal{K} but 2^n in K .

Theorem 3.8. $\sigma_\infty \subset \pi_\infty^{\mathcal{S}}$, so that $\pi_\infty^{\mathcal{S}}$ is dense in Σ_π and $\Sigma_\pi := \overline{\pi_\infty} = \overline{\sigma_\infty} = \overline{\pi_\infty^{\mathcal{S}}}$.

Proof. We will show in Proposition 3.9 below that, for $n \geq 2$,

$$\sigma_n \subset \pi_{2n+2}^{\mathcal{S}} := \bigcup_{k \in \mathcal{K}_{2n+2}} \text{spec } A_k^{\text{per}} \subset \pi_{2n+2},$$

where, for $m \geq 2$, \mathcal{K}_m denotes the set of those vectors in \mathcal{K} that have length m . Since also $\sigma_1 = \{0\} \subset \pi_m^{\mathcal{S}}$, for every $m \in \mathbb{N}$ (e.g. [4, Lemma 2.10]), this implies that $\sigma_\infty \subset \pi_\infty^{\mathcal{S}}$, which implies that $\pi_\infty^{\mathcal{S}}$ is dense in Σ_π since σ_∞ is dense in π_∞ [16, Theorem 1], and the result follows. \square

The key step in the proof of the above theorem is the following refinement of Theorem 4.1 in [4], which uses our new characterisation, Proposition 3.3, of \mathcal{S} .

Proposition 3.9. Suppose $a, b, c, d \in \{\pm 1\}$ and $k \in \{\pm 1\}^n$, for some $n \geq 2$, and let $\tilde{k} := (k_1, \dots, k_{n-1})$. Then

$$\text{spec } A_k^{(n)} \subset \text{spec } A_\ell^{\text{per}}, \quad \text{for } \ell = (\tilde{k}', a, b, \tilde{k}, c, d) \in \{\pm 1\}^{2n+2}, \quad (3.9)$$

where $\tilde{k}' = \tilde{k}J_{n-1} = (k_{n-1}, \dots, k_1)$. Further, $\text{spec } A_k^{(n)} \subset \pi_{2n+2}^{\mathcal{S}}$.

Proof. The proof modifies [4, Theorem 4.1] where the same result is proved for the special case that $a = c = -1$, $b = d = 1$. Following that proof, suppose that λ is an eigenvalue of $A_k^{(n)}$ with corresponding eigenvector x , let $\hat{x} := J_n x$ and $\hat{A}_k^{(n)} := J_n A_k^{(n)} J_n$, so that $\hat{A}_k^{(n)} \hat{x} = \lambda \hat{x}$, and set

$$B := \begin{pmatrix} \ddots & & & & & & \\ & d & & & & & \\ & & \boxed{1} & & & & \\ & & \boxed{\hat{A}_k^{(n)}} & & & & \\ & & & -1 & & & \\ & & & & \boxed{0} & & \\ & & & & & b & \\ & & & & & & 1 \\ & & & & & & \boxed{A_k^{(n)}} \\ & & & & & & & -1 \\ & & & & & & & & -c \\ & & & & & & & & 0 \\ & & & & & & & & & d \\ & & & & & & & & & & 1 \\ & & & & & & & & & & \boxed{\hat{A}_k^{(n)}} \\ & & & & & & & & & & & -a \\ & & & & & & & & & & & & -1 \\ & & & & & & & & & & & & & \ddots \end{pmatrix},$$

where $\boxed{0}$ marks the entry at position $(-1, -1)$. Then B is a bi-infinite tridiagonal matrix with zeros on the main diagonal, and with each of the first sub- and super-diagonals a vector in Ω that is periodic with period $2n + 2$. Define $\tilde{x} \in \ell^\infty$, the space of bounded, complex-valued sequences $\phi : \mathbb{Z} \rightarrow \mathbb{C}$, by

$$\tilde{x} := (\dots, 0, \hat{x}^T, \boxed{0}, x^T, 0, \hat{x}^T, 0, x^T, \dots)^T,$$

where $\boxed{0}$ marks the entry \tilde{x}_{-1} . Then it is easy to see that $B\tilde{x} = \lambda\tilde{x}$, so that $\lambda \in \text{spec } B$.⁴ Further, by a simple gauge transformation [4, Lemma 3.2], $\text{spec } B = \text{spec } A_\ell^{\text{per}}$, where ℓ is given by (3.9).

We have shown that $\text{spec } A_k^{(n)} \subset \text{spec } A_\ell^{\text{per}}$. But, choosing in particular $a = b$ and $c = -1$, $d = 1$, we see from Proposition 3.3 that $\ell \in \mathcal{K}_{2n+2}$, so that $\text{spec } A_k^{(n)} \subset \text{spec } A_\ell^{\text{per}} \subset \pi_{2n+2}^{\mathcal{S}}$. \square

Remark 3.10. Proposition 3.9 implies that $\text{spec } A_k^{(n)} \subset \text{spec } A_\ell^{\text{per}}$ for 16 different vectors $\ell \in \{\pm 1\}^n$, corresponding to different choices of $a, b, c, d \in \{\pm 1\}$. Some of these vectors correspond to the same polynomial p_ℓ and hence, by (2.8), to the same spectrum $\text{spec } A_\ell^{\text{per}}$. In particular, if $a = b = \pm 1$, then the choices $c = -d = 1$ and $c = -d = -1$ lead to the same polynomial by Proposition 3.3 and the definition of \mathcal{S} . But, if $a \neq b$, again by Proposition 3.3 and the definition of \mathcal{S} , the choices $c = -d = 1$ and $c = -d = -1$ must lead to different polynomials, and neither of these polynomials can be in \mathcal{S} . On the other hand, as observed already in the proof of Proposition 3.9, the choices $a = b$ and $c = -d$ lead to an $\ell \in \mathcal{K}$

⁴Clearly λ is in the spectrum of B as an operator on ℓ^∞ , but this is the same as the ℓ^2 -spectrum from general results on band operators, e.g. [20].

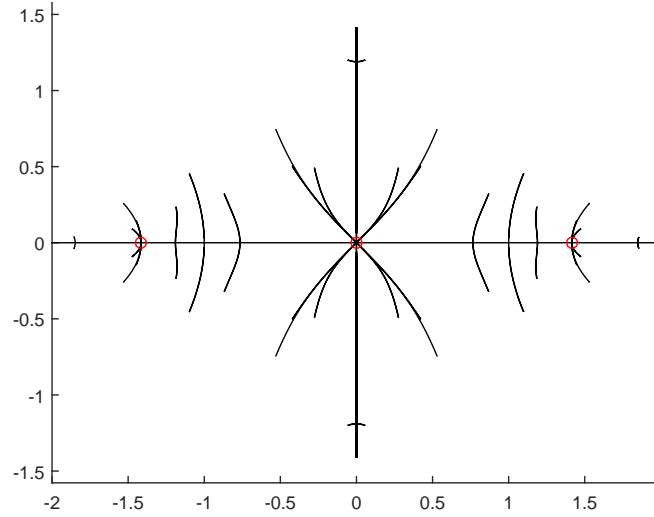


FIGURE 1. An illustration of Proposition 3.9 in the case $n = 3$, $k_1 = k_2 = 1$. The red circles indicate the eigenvalues, 0 and $\pm\sqrt{2}$, of $A_k^{(3)}$. The black lines are the spectra of A_ℓ^{per} , for the different choices of ℓ defined by (3.9). In this case there are 7 distinct polynomials p_ℓ and 7 associated distinct spectra $\text{spec } A_\ell^{\text{per}}$, each of which contains the eigenvalues of $A_k^{(3)}$. One cannot see all the spectra as separate curves because some of them overlap.

and so to a polynomial $p_\ell \in \mathcal{S}$. Thus Proposition 3.9 implies that there are at least three distinct polynomials p_ℓ such that $\text{spec } A_k^{(n)} \subset \text{spec } A_\ell^{\text{per}}$. Figure 1 plots $\text{spec } A_k^{(n)}$ and $\text{spec } A_\ell^{\text{per}}$, for the vectors defined by (3.9), in the case that $n = 3$ and $k_1 = k_2 = 1$. For other plots of the spectra of finite and periodic Feinberg-Zee matrices, and the interrelation of these spectra, see [19, 3, 4].

3.2. Explicit lower bounds for Σ_π

As noted in [17], that the polynomials $p \in \mathcal{S}$ satisfy (3.3) gives us a tool to compute explicit lower bounds on Σ_π . Indeed [17] shows visualisations of $p^{-n}(\mathbb{D})$, for several $p \in \mathcal{S}$ and $n \in \mathbb{N}$, and visualisations of unions of $p^{-n}(\mathbb{D})$ for p varying over some finite subset of \mathcal{S} . Since, by (1.5), $\mathbb{D} \subset \Sigma_\pi$, it follows from (3.3) that all these sets are subsets of Σ_π .

The study in [17] contains visualisations of subsets of Σ_π as just described, but no associated analytical calculations. Complementing the study in [17] we make explicit calculations in this section that illustrate the use of the polynomial symmetries to compute explicit formulae for regions of the complex plane that are subsets of Σ_π , adding to the already known fact (1.5) that $\mathbb{D} \subset \Sigma_\pi$.

Our first lemma and corollary give explicit formulae for $p^{-1}(\mathbb{D})$ when $p(\lambda) = P_3(\lambda) = \lambda^3 - \lambda$ and when $p(\lambda) = Q_2(\lambda) = \lambda^3 + \lambda$. These formulae are expressed in terms of the function $s \in C^\infty[-1, 1]$, where, for $-1 \leq t \leq 1$, $s(t) > 0$ is defined as the largest positive solution of

$$f(s) := s^3 - 2ts^2 + s = 1.$$

If $-1 \leq t < 1/2$, this is the unique solution in $(0, 1)$ (on which interval $f'(s) \geq 0$), while if $1/2 \leq t < 1$ it is the unique solution in $[1, 2)$ (on which interval $f'(s) \geq 0$). Explicitly, for $-1 \leq t \leq 1$,

$$s'(t) = \frac{2(s(t))^3}{2 - s(t) + (s(t))^3} > 0,$$

and $s(1/2) = 1$, $s(1) \approx 1.75488$.

Lemma 3.11. *If $p(\lambda) := P_3(\lambda) = \lambda^3 - \lambda$, then*

$$p^{-1}(\mathbb{D}) = E := \{re^{i\theta} : 0 \leq r < S(\theta), 0 \leq \theta < 2\pi\},$$

where $S(\theta) := \sqrt{s(\cos 2\theta)}$, for $\theta \in \mathbb{R}$. $S \in C^\infty(\mathbb{R})$ and is even and periodic with period π . In the interval $[0, \pi]$ the only stationary points of S are global maxima at 0 and π , with $S(0) = S(\pi) = \sqrt{s(1)} \approx 1.32472$, and a global minimum at $\pi/2$. Further, S is strictly decreasing on $[0, \pi/2]$ and $S(\theta) \geq 1$ in $[0, \pi]$ iff $0 \leq \theta \leq \pi/6$ or $5\pi/6 \leq \theta \leq \pi$, with equality iff $\theta = \pi/6$ or $5\pi/6$.

Proof. If $\lambda = re^{i\theta}$ (with $r \geq 0$, $\theta \in \mathbb{R}$), then $|p(\lambda)|^2 = |\lambda^3 - \lambda|^2 = |r^3 e^{2i\theta} - r|^2 = r^6 - 2r^4 \cos(2\theta) + r^2$. It is straightforward to show that $|p(\lambda)| < 1$ iff $0 \leq r < S(\theta)$. The properties of S claimed follow easily from the properties of s stated above. \square

Corollary 3.12. *If $p(\lambda) := Q_2(\lambda) = \lambda^3 + \lambda$, then*

$$p^{-1}(\mathbb{D}) = iE = \{re^{i\theta} : 0 \leq r < S(\theta - \pi/2), 0 \leq \theta < 2\pi\}.$$

Proof. This is clear from Lemma 3.11 and the observation that $Q_2(\lambda) = iP_3(i\lambda)$. \square

Since $P_3, Q_2 \in \mathcal{S}$, it follows from the above lemma and corollary and (3.4) that $E \cup iE \subset \text{int}(\Sigma_\pi)$. But this implies by (3.4), since $P_2(\lambda) = \lambda^2$ is also in \mathcal{S} , that also $\sqrt{iE} \subset \text{int}(\Sigma_\pi)$, where, for $S \subset \mathbb{C}$, $\sqrt{S} := \{\lambda \in \mathbb{C} : \lambda^2 \in S\}$. In particular,

$$W_1 := \{re^{i\theta} : 0 \leq r < S(\theta), -\pi/6 \leq \theta \leq \pi/6\} \subset E \subset \text{int}(\Sigma_\pi) \quad \text{and} \quad (3.10)$$

$$W_2 := \{re^{i\theta} : 0 \leq r < \sqrt{S(2\theta - \pi/2)}, \pi/6 \leq \theta \leq \pi/3\} \subset \sqrt{iE} \subset \text{int}(\Sigma_\pi). \quad (3.11)$$

It is easy to check that

$$\bigcup_{m=0}^3 i^m(W_1 \cup W_2) = E \cup iE \cup \sqrt{iE} \subset \text{int}(\Sigma_\pi). \quad (3.12)$$

Next note from Table 1 that $p \in \mathcal{S}$ where $p(\lambda) := \lambda^5 - \lambda^3 + \lambda$ factorises as

$$p(\lambda) = \lambda(\lambda - e^{i\pi/6})(\lambda + e^{i\pi/6})(\lambda - e^{-i\pi/6})(\lambda + e^{-i\pi/6}).$$

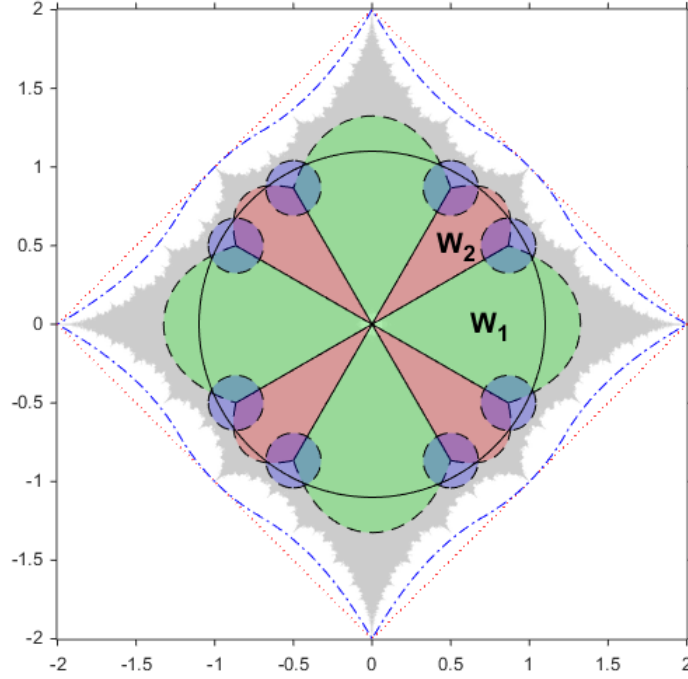


FIGURE 2. A plot showing W_1 (green), W_2 (red), $e^{\pm i\pi/6} + \eta\mathbb{D}$ (blue), and their rotations by multiples of $\pi/2$. The union of the green, red, and blue regions is $W \subset \Sigma_\pi$, defined by (3.14). W contains $1.1\overline{\mathbb{D}}$, indicated by the black circle, see Proposition 3.13. In the background in grey one can see $\pi_{30} \subset \Sigma_\pi$. The dotted and dashed-dotted curves are the boundaries of $\overline{\Delta}$ and N_2 , respectively, defined by (1.3) and (2.12), with $\overline{\Delta} \supset N_2 \supset \Sigma \supset \Sigma_\pi$.

Thus, for $\lambda = \exp(\pm i\pi/6) + w$ with $|w| \leq \epsilon$,

$$|p(\lambda)| \leq (1+\epsilon)\epsilon(2+\epsilon)(2\sin(\pi/6)+\epsilon)(2\cos(\pi/6)+\epsilon) = \epsilon(1+\epsilon)^2(\sqrt{3}+\epsilon)(2+\epsilon) =: g(\epsilon).$$

Let $\eta \approx 0.174744$ be the unique positive solution of $g(\epsilon) = 1$. Clearly $|p(\lambda)| < 1$ if $\lambda = \exp(\pm i\pi/6) + w$, with $|w| < \eta$, so that

$$\exp(\pm i\pi/6) + \eta\mathbb{D} \subset p^{-1}(\mathbb{D}) \subset \text{int}(\Sigma_\pi). \quad (3.13)$$

We have shown most of the following proposition that extends to a region $W \supset 1.1\overline{\mathbb{D}}$ (illustrated in Figure 2) the part of the complex plane that is known to consist of interior points of Σ_π , making explicit implications of the polynomial symmetries \mathcal{S} of Σ . Before [17] and the current paper the most that was known explicitly was that $\mathbb{D} \subset \text{int}(\Sigma_\pi)$.

Proposition 3.13.

$$1.1\overline{\mathbb{D}} \subset W := \bigcup_{m=0}^3 i^m \left(W_1 \cup W_2 \cup (e^{i\pi/6} + \eta\mathbb{D}) \cup (e^{-i\pi/6} + \eta\mathbb{D}) \right) \subset \text{int}(\Sigma_\pi). \quad (3.14)$$

Proof. That $W \subset \text{int}(\Sigma_\pi)$ is (3.12) combined with (3.13) and Lemma 3.1. It is easy to see that W is invariant with respect to the maps $\lambda \mapsto i\lambda$ and $\lambda \mapsto \overline{\lambda}$, and so is invariant under the dihedral symmetry group D_2 generated by these two maps. Thus to complete the proof it is sufficient to show that $z := re^{i\theta} \in W$ for $0 \leq r \leq 1.1$ and $0 \leq \theta \leq \pi/4$.

Now since, by Lemma 3.11, $S(\theta) \geq 1$ for $-\pi/6 \leq \theta \leq \pi/6$, $z \in W_1 \cup W_2$ for $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/4$. Further, since, by Lemma 3.11, S is even and is increasing on $[-\pi/6, 0]$, $S(\theta) \geq S(\pi/8) = \sqrt{s(\sqrt{2}/2)}$, for $0 \leq \theta \leq \pi/8$, and $\sqrt{S(2\theta - \pi/2)} \geq \sqrt{S(\pi/12)} = (s(\sqrt{3}/2))^{1/4}$, for $5\pi/24 \leq \theta \leq \pi/4$. But $\sqrt{3} > 1.73$, so that $f(1.5) < 0.9825$ for $t = \sqrt{3}/2$, which implies that $s(\sqrt{3}/2) > 1.5$, so that $(s(\sqrt{3}/2))^{1/4} > 1.1$. Similarly, $\sqrt{2}/2 > 0.705$ and $s(0.705) = 1.25$, so that $\sqrt{s(\sqrt{2}/2)} > \sqrt{1.25} > 1.1$. Thus $z \in W_1$ for $0 \leq r \leq 1.1$ and $0 \leq \theta \leq \pi/8$, while $z \in W_2$ for $0 \leq r \leq 1.1$ and $5\pi/24 \leq \theta \leq \pi/4$.

To conclude that $1.1\overline{\mathbb{D}} \subset W$, it remains to show that $z \in W$ for $1 \leq r \leq 1.1$ and $|\pi/6 - \theta| \leq \pi/24$. But it is easy to check that, for these ranges of r and θ , $z \in \exp(i\pi/6) + \eta\mathbb{D}$ provided $\cos(\pi/24) + \sqrt{\cos^2(\pi/24) + \eta^2 - 1} > 1.1$. But this last inequality holds since $\cos(\pi/24) = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{3}}} > 0.991$ and $g(0.174) < 1$ so that $\eta > 0.174$. \square

4. Interior points of Σ_π

We have just, in Proposition 3.13, extended to a region $W \supset 1.1\overline{\mathbb{D}}$ the part of the complex plane that is known to consist of interior points of Σ_π . In this section we explore the relationship between Σ_π and its interior further. We show first of all, using (3.4) and that $P_n \in \mathcal{S}$ for every $n \geq 2$, that $[0, 2) \subset \text{int}(\Sigma_\pi)$. Next we use this result to show that, for every $n \geq 2$, all but finitely many points in $\pi_n^{\mathcal{S}}$ are interior points of Σ_π . Finally, we prove, using Theorem 3.8, that Σ_π is the closure of its interior. If indeed it can be shown, as conjectured in [3], that $\Sigma_\pi = \Sigma$, then the result will imply the truth of another conjecture in [3], that Σ is the closure of its interior.

Our technique for establishing that $[0, 2) \subset \text{int}(\Sigma_\pi)$ will be to use that $P_m^{-1}((-1, 1)) \subset \text{int}(\Sigma_\pi)$, for every $m \geq 3$, this a particular instance of (3.4). This requires first a study of the real solutions of the equations $P_m(\lambda) = \pm 1$ and their interlacing properties, which we now undertake.

From (3.6),

$$P_m(2) = 2m, \quad P_m(2 \cos(\pi/m)) = 0, \quad \text{and} \quad P_m(2 \cos(3\pi/(2m))) = -2 \cot(3\pi/(2m)).$$

This implies that the equation $P_m(\lambda) = 1$ has a solution in $(2 \cos(\pi/m), 2)$. Let λ_m^+ denote the largest solution in this interval. Further, if $m \geq 5$ then $P_m(\lambda) = -1$ has a solution in $(2 \cos(3\pi/(2m)), 2)$ since $-2 \cot(3\pi/(2m)) < -1$. For $m \geq 4$ let λ_m^- denote the largest solution to $P_m(\lambda) = -1$ in $(0, 2)$, which is in the interval $(2 \cos(3\pi/(2m)), 2)$ if $m \geq 5$, while an explicit calculation gives that $\lambda_4^- = 1$.

Throughout the following calculations we use the notation $r_n(\theta)$ from (3.6).

Lemma 4.1. *For $m \geq 4$ it holds that P_m is strictly increasing on $(\lambda_m^-, 2)$, that $\lambda_m^- < \lambda_m^+$, and that $-1 < P_m(\lambda) < 1$ for $\lambda_m^- < \lambda < \lambda_m^+$.*

Proof. Explicitly, $P_4(\lambda) = \lambda U_3(\lambda/2) = \lambda^4 - 2\lambda^2$, so that $P_4'(\lambda) = 4(\lambda^3 - 1)$ and these claims are clear for $m = 4$.

Suppose now that $m \geq 5$. It follows by induction that, for $n \geq 3$, $r_n(\theta)$ is strictly decreasing on $(0, \pi/n + \pi/n^2)$. For $r_3(\theta) = 2 \cos \theta (4 \cos^2 \theta - 1)$ is strictly decreasing on $(0, 4\pi/9) \subset (0, \pi/2)$ and, if this statement is true for some $n \geq 3$, then

$$r_{n+1}(\theta) = 2 \cot \theta \sin((n+1)\theta) = \cos \theta (r_n(\theta) + 2 \cos n\theta)$$

is strictly decreasing on $(0, \pi/(n+1) + \pi/(n+1)^2) \subset (0, \pi/n)$. Further,

$$\begin{aligned} r_m(\pi/m + \pi/m^2) &= -2 \cos(\pi/m + \pi/m^2) \sin(\pi/m) / \sin(\pi/m + \pi/m^2) \\ &< -\frac{2m \cos(\pi/5 + \pi/25)}{m+1} < -10 \cos(6\pi/25)/6 < -1, \end{aligned}$$

since $\sin a / \sin b > a/b$ for $0 < a < b < \pi$. As $r_m(\theta) = P_m(2 \cos \theta)$, these observations imply that, on $(2 \cos(\pi/m + \pi/m^2), 2)$, P_m is strictly increasing, and that $\lambda_m^- > 2 \cos(\pi/m + \pi/m^2)$. Thus $\lambda_m^+ > \lambda_m^-$ follows from the definitions of λ_m^\pm , and $-1 < P_m(\lambda) < 1$ for $\lambda_m^- < \lambda < \lambda_m^+$. \square

As observed above it follows from (3.4) that $P_m^{-1}((-1, 1)) \subset \text{int}(\Sigma_\pi)$, for $m \geq 2$. Combining this observation with Lemma 4.1 and the fact that $P_3(\lambda) = \lambda^3 - \lambda \in (-1, 1)$ if $-\lambda_3^+ < \lambda < \lambda_3^+$, we obtain the following corollary.

Corollary 4.2. *For $m = 4, 5, \dots$, $(\lambda_m^-, \lambda_m^+) \subset \text{int}(\Sigma_\pi)$. Also, $(-\lambda_3^+, \lambda_3^+) \subset \text{int}(\Sigma_\pi)$.*

By definition of λ_m^\pm , $\lambda_m^\pm \rightarrow 2$ as $m \rightarrow \infty$. Thus the above corollary and the following lemma together imply that $[0, 2) \subset \text{int}(\Sigma_\pi)$.

Lemma 4.3. *For $m = 3, 4, \dots$, $\lambda_{m+1}^- < \lambda_m^+ < \lambda_{m+1}^+$.*

Proof. Since $P_3(\lambda) = \lambda^3 - \lambda$ and $P_4(\lambda) = \lambda^4 - 2\lambda^2$, it is easy to see that $1 = \lambda_4^- < \lambda_3^+ < \lambda_4^+$, so that the claimed result holds for $m = 3$.

To see the result for $m \geq 4$ we will show, equivalently, that $r_{m+1}^- > r_m^+ > r_{m+1}^+$, for $m = 4, 5, \dots$. Here r_n^+ , for $n \in \mathbb{N}$, is the smallest solution of $r_n(\theta) = 1$ in $(0, \pi/n)$, so that $\lambda_n^+ = 2 \cos r_n^+$, while r_n^- , for $n = 4, 5, \dots$, denotes the smallest solution of $r_n(\theta) = -1$ in $(0, \pi/2)$, so that $\lambda_n^- = 2 \cos r_n^-$.

We have shown in the proof of Lemma 4.1 that, for $m \geq 3$, $r_m(\theta)$ is strictly decreasing on $(0, \pi/m + \pi/m^2)$, and that $r_m(\pi/m + \pi/m^2) < -1$, for $m \geq 5$, while $r_m(\pi/m) = 0$ so that

$$\frac{\pi}{m} < r_m^- < \frac{\pi}{m} + \frac{\pi}{m^2}, \quad \text{for } m \geq 5.$$

Similarly, for $m \geq 2$,

$$\begin{aligned} r_m(\pi/m - \pi/m^2) &= 2 \cos(\pi/m - \pi/m^2) \sin(\pi/m) / \sin(\pi/m - \pi/m^2) \\ &> 2 \cos(\pi/2 - \pi/4) = \sqrt{2} > 1, \end{aligned}$$

so that

$$\frac{\pi}{m} - \frac{\pi}{m^2} < r_m^+ < \frac{\pi}{m}, \quad \text{for } m \geq 2.$$

These inequalities imply that, for $m \geq 4$, $r_{m+1}^- \in (0, \pi/m + \pi/m^2)$, and since r_m is strictly decreasing on this interval and

$$\begin{aligned} r_m(r_{m+1}^-) &= 2 \cot r_{m+1}^- \sin((m+1)r_{m+1}^- - r_{m+1}^-) \\ &= \cos r_{m+1}^- (-1 - 2 \cos((m+1)r_{m+1}^-)) < \cos r_{m+1}^- < 1, \end{aligned}$$

it follows that $r_m^+ < r_{m+1}^- < \pi/(m+1) + \pi/(m+1)^2$. Since also r_{m+1} is strictly decreasing on $(0, \pi/(m+1) + \pi/(m+1)^2)$ and

$$r_{m+1}(r_m^+) = 2 \cot r_m^+ \sin(mr_m^+ + r_m^+) = \cos r_m^+ (1 + 2 \cos(mr_m^+)) < \cos r_m^+ < 1,$$

since $\pi/2 < mr_m^+ < \pi$, we see that also $r_m^+ > r_{m+1}^+$. \square

Corollary 4.4. $(-2, 2) \cup i(-2, 2) \subset \text{int}(\Sigma_\pi)$.

Proof. From Corollary 4.2 and Lemma 4.3 it follows that $[0, \lambda_3^+) \subset \text{int}(\Sigma_\pi)$ and that $[\lambda_m^+, \lambda_{m+1}^+) \subset \text{int}(\Sigma_\pi)$, for $m \geq 3$. Thus $[0, \lambda_m^+) \subset \text{int}(\Sigma_\pi)$ for $m \geq 3$, and so $[0, 2) \subset \text{int}(\Sigma_\pi)$ since $\lambda_m^+ \rightarrow 2$ as $m \rightarrow \infty$. Applying Lemma 3.1 we obtain the stated result. \square

The following lemma follows immediately from Corollary 4.4, (2.8), and (3.4).

Lemma 4.5. *Suppose that $k \in \mathcal{K}_n$, so that k has length n and $p_k \in \mathcal{S}$ has degree n . Then all except at most $2n$ points in $\text{spec } A_k^{\text{per}}$ are interior points of Σ_π . Further, if $\lambda \in \text{spec } A_k^{\text{per}}$ then there exists a sequence $(\lambda_m) \subset \text{spec } A_k^{\text{per}} \cap \text{int}(\Sigma_\pi)$ such that $\lambda_m \rightarrow \lambda$ as $m \rightarrow \infty$.*

As an example of the above lemma, suppose that $k = (-1, 1) \in \mathcal{K}_2$. Then (see Table 1) $p_k(\lambda) = \lambda^2$ and, from (2.8), $\text{spec } A_k^{\text{per}} = \{x \pm ix : -1 \leq x \leq 1\}$. There are precisely four points, $\pm 1 \pm i \in \text{spec } A_k^{\text{per}} \setminus \text{int}(\Sigma_\pi)$. These are not interior points of Σ_π since they lie on the boundary of $\bar{\Delta} \supset \Sigma \supset \Sigma_\pi$.

Combining the above lemma with Theorem 3.8, we obtain the last result of this section.

Theorem 4.6. Σ_π is the closure of its interior.

Proof. Suppose $\lambda \in \Sigma_\pi$. Then, by Theorem 3.8, λ is the limit of a sequence $(\lambda_n) \subset \pi_\infty^S$, and, by Lemma 4.5, for each n there exists $\mu_n \in \text{int}(\Sigma_\pi)$ such that $|\mu_n - \lambda_n| < n^{-1}$, so that $\mu_n \rightarrow \lambda$ as $n \rightarrow \infty$. \square

5. Filled Julia sets in Σ_π

It was shown in [17] that, for every polynomial symmetry $p \in \mathcal{S}$, the corresponding Julia set $J(p)$ satisfies $J(p) \subset \overline{U(p)} \subset \Sigma_\pi$, where $U(p)$ is defined by (1.7). (The argument in [17] is that $J(p) \subset \overline{U(p)}$ by (2.10), and that $\overline{U(p)} \subset \Sigma_\pi$ by (3.4).) It was conjectured in [17] that also the filled Julia set $K(p) \subset \overline{U(p)} \subset \Sigma_\pi$, for every $p \in \mathcal{S}$. In this section we will first show by a counterexample that this conjecture is false; we will exhibit a $p \in \mathcal{S}$ of degree 18 for which $K(p) \not\subset \overline{U(p)}$. However, we have no reason to doubt a modified conjecture, that $K(p) \subset \Sigma_\pi$, for all $p \in \mathcal{S}$. And the main result of this section will be to prove that $K(p) \subset \Sigma_\pi$ for a large class of $p \in \mathcal{S}$, including $p = P_m$, for $m \geq 2$.

Our first result is the claimed counterexample.

Lemma 5.1. *Let $k = (1, -1, 1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, -1, 1)$, so that (by Proposition 3.3) $p_k \in \mathcal{S}$, this polynomial given explicitly by*

$$p_k(\lambda) = \lambda^{18} - 4\lambda^{16} + 5\lambda^{14} - 4\lambda^{12} + 7\lambda^{10} - 8\lambda^8 + 6\lambda^6 - 4\lambda^4 + \lambda^2.$$

Then $K(p_k) \not\subset \overline{U(p_k)}$.

Proof. Let $p = p_k$. If we can find a $\mu \notin \overline{\mathbb{D}}$ that is an attracting fixed point of p , then, for all sufficiently small $\epsilon > 0$, $N := \mu + \epsilon\mathbb{D}$ satisfies $p(N) \subset N$ and $N \cap \overline{\mathbb{D}} = \emptyset$, so that $N \subset K(p)$ and $N \cap \overline{U(p)} = \emptyset$. Calculating in double-precision floating-point arithmetic in Matlab we see that $\lambda \approx 1.21544069$ appears to be a fixed point of p , with

$$p'(\lambda) = 18\lambda^{17} - 64\lambda^{15} + 70\lambda^{13} - 48\lambda^{11} + 70\lambda^9 - 64\lambda^7 + 36\lambda^5 - 16\lambda^3 + 2\lambda \approx -0.69,$$

so that this fixed point appears to be attracting. To put this on a rigorous footing we work in exact arithmetic to deduce, by the intermediate value theorem, that $p(\lambda) = \lambda$ has a solution $\lambda^* \in (1.215, 1.216)$, and that $|p'(1.2155)| \leq 0.71$. Then, noting that $p''(\lambda) = p_+(\lambda) - p_-(\lambda)$, where $p_+(\lambda) = 306\lambda^{16} + 910\lambda^{12} + 630\lambda^8 + 180\lambda^4 + 2$ and $p_-(\lambda) = 960\lambda^{14} + 528\lambda^{10} + 448\lambda^6 + 48\lambda^2$, we see that

$$|p''(\lambda)| \leq \max\{|p_-(1.216) - p_+(1.215)|, |p_-(1.215) - p_+(1.216)|\} < 400$$

for $1.215 \leq \lambda \leq 1.216$. But this implies that $|p'(\lambda^*)| \leq |p'(1.2155)| + 0.0005 \times 400 \leq 0.91$, so that λ^* is an attracting fixed point. \square

Numerical results suggest that amongst the polynomials $p \in \mathcal{S}$ of degree ≤ 20 , there is only one other similar counterexample of a polynomial with an attracting fixed point outside the unit disk, the other example of degree 19.

We turn now to positive results. Part of our argument will be to show, for every $p \in \mathcal{S}$, that $\{z : |z| \geq 2\} \subset A_p(\infty)$, via the following lower bounds that follow immediately from Lemma 2.3, (2.6) and (3.5).

Corollary 5.2. *If $k \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$, then $|p_k(\lambda)| \geq 2$, for $|\lambda| \geq 2$. If $p_k \in \mathcal{S}$, then $|p_k(\lambda)| \geq 2n$, for $|\lambda| \geq 2$.*

Corollary 5.3. *Let $p = p_k$, where $k \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$. Then $A_p(\infty) \supset \{z \in \mathbb{C} : |z| > 2\}$. If $p \in \mathcal{S}$, then $A_p(\infty) \supset \{z \in \mathbb{C} : |z| \geq 2\}$.*

Proof. Let $z \in \mathbb{C}$ with $|z| > 2$. Then, by Corollary 5.2, for some neighbourhood N of z , $|p_k(w)| \geq 2$ for $w \in N$. Thus, and by Montel's theorem [11, Theorem 14.5], the family $\{p^n : n \in \mathbb{N}\}$ is normal at z . So $z \notin J(p)$, by (2.9). We have shown that $J(p) \cap \{z : |z| > 2\} = \emptyset$, so that also $K(p) \cap \{z : |z| > 2\} = \emptyset$ and $A_p(\infty) = \mathbb{C} \setminus K(p) \supset \{z : |z| > 2\}$.

If $p \in \mathcal{S}$ and $|z| = 2$ then, by Corollary 5.2, $|p(z)| \geq 4$ so that $p(z) \in A_p(\infty)$ and so $z \in A_p(\infty)$. Thus $A_p(\infty) \supset \{z : |z| \geq 2\}$. \square

We remark that the bounds in Corollary 5.2 appear to be sharp. In particular, if $k = (1, 1, \dots, 1)$ has length $n \geq 2$, we see from (2.5), (3.5), and Lemma 3.5 that $p_k(2) = U_n(1) - U_{n-2}(1) = 2$, since $U_m(1) = m + 1$ [1]. And we note that, if $p = P_m$, for some $m \in \mathbb{N}$, then $p(2) = P_m(2) = 2U_{m-1}(1) = 2m$. Finally, we recall that we have already noted that, for $p = p_k$, with $k = (1, 1)$, i.e., $p(z) = z^2 - 2$, the Julia set is $J(p) = [-2, 2]$, so that $A_p(\infty) \not\supset \{z : |z| \geq 2\}$ for this p .

The polynomial $p(z) = z^2 - 2$ is an example where $J(p) = K(p)$ so $F_B(p) = K(p) \setminus J(p) = \emptyset$. The next lemma tells us that this does not happen, that $K(p)$ is strictly larger than $J(p)$, if $p \in \mathcal{S}$.

Lemma 5.4. *$F_B(p) \cap U(p)$ is non-empty for $p \in \mathcal{S}$.*

Proof. If $p \in \mathcal{S}$ is even then, by Lemma 2.5 and Corollary 3.4, $p(0) = p'(0) = 0$, so that 0 is an attracting fixed point. Clearly $A_p(0)$ (which is non-empty) is a subset of $U(p) \cap F_B(p)$. Similarly, by Lemma 2.5 and Corollary 3.4, if p is odd then $p(0) = 0$ and $p'(0) = \pm 1$, so that $0 \in J(p)$ is a rationally neutral fixed point and has a (non-empty) attracting region contained in $F_B(p)$ [2, Section II.5], this region clearly also in $U(p)$. \square

The above lemma and (3.4) imply that $F_B(p) \cap \Sigma_\pi \supset F_B(p) \cap U(p)$ is non-empty for all $p \in \mathcal{S}$, in particular that $A_p(0) \subset F_B(p) \cap U(p) \subset \Sigma_\pi$ if p is even. The main result of this section is the following criterion for the *whole* of $F_B(p)$ to be contained in Σ_π .

Theorem 5.5. *Suppose that $p \in \mathcal{S}$, and that the critical points of p in $K(p)$ have orbits that lie eventually in $1.1\overline{\mathbb{D}} \cup (-2, 2) \cup i(-2, 2)$. Then $K(p) \subset \Sigma_\pi$.*

Proof. Choose a and b with $-2 < a < b < 2$ such that $K(p) \cap \mathbb{R} \subset [a, b]$ and $K(p) \cap i\mathbb{R} \subset i[a, b]$, this possible by Corollary 5.3 which says that the closed set $K(p) \subset \{z : |z| < 2\}$. Set $T = [a, b] \cup i[a, b] \cup 1.1\overline{\mathbb{D}}$, and choose a simply-connected open set S such that $T \subset S \subset \Sigma_\pi$, this possible by Corollary 4.4 and Proposition 3.13. By hypothesis, the orbits of the critical points in $K(p)$ lie eventually in T . Thus the lemma follows from Proposition 2.11 and (3.4). \square

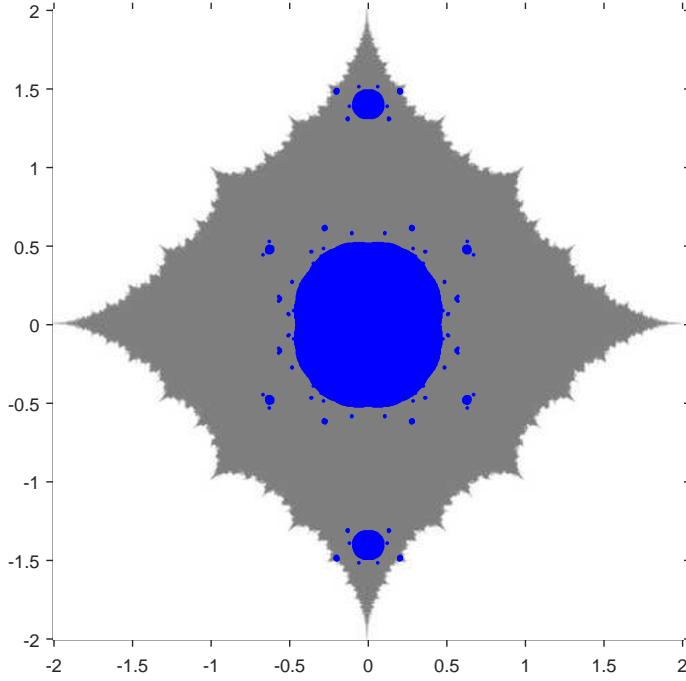


FIGURE 3. Plot of $\pi_{30} \subset \Sigma_\pi$ and (in blue) the filled Julia set $K(p)$ in the case $p(\lambda) = P_4^*(\lambda) = \lambda^4 + 2\lambda^2$. By Corollary 5.6, $K(p) \subset \Sigma_\pi$.

As an example of application of this theorem, consider $p \in \mathcal{S}$ given by (see Table 1) $p(\lambda) = P_4^*(\lambda) = \lambda^4 + 2\lambda^2$. This p has critical points 0 and $\pm i$. Since $p^2(\pm i) = 3$ it follows from Corollary 5.3 that $\pm i \in A_p(\infty)$, while 0 is a fixed point. Theorem 5.5 tells us that $K(p)$, visualised in Figure 3, is contained in Σ_π . We note that, since all the critical points of p except the fixed point 0 are in $A_p(\infty)$, $K(p)$ is not connected [2, Theorem III 4.1] and, by Theorem 2.8 and Proposition 2.10, $F_B(p) = A_p(0)$, which implies that $K(p) \subset \overline{U(p)}$. Further, recalling the discussion in Section 2, $J(p) = \partial K(p) = \partial A_p(0) = \partial A_p(\infty)$, and, since $K(p)$ has more than one component, $F_B(p)$ has infinitely many components [2, Theorem IV 1.2].

The above example is a particular instance of a more general result. It is straightforward to see that if p is a polynomial with zeros only on the real line, then all the critical points are also on the real line. Since, by Lemma 3.5, $P_m(\lambda) = \lambda U_{m-1}(\lambda/2)$, and all the zeros of the polynomial U_{m-1} are real, it follows that all the zeros of P_m are real, so all its critical points are also real, and so the orbits of all the critical points are real. Further, by Corollary 5.3, the orbits of the critical points in $K(p)$ stay in $(-2, 2)$. Likewise, as (see (3.7)) $P_m^*(\lambda) = i^{-m} P_m(i\lambda)$, all the critical points of P_m^* lie on $i\mathbb{R}$, and so the orbits of these critical points are real if

m is even, pure imaginary if m is odd. Further, by Corollary 5.3, the orbits of the critical points in $K(p)$ stay in $(-2, 2) \cup i(-2, 2)$. Applying Theorem 5.5 we obtain:

Corollary 5.6. $K(P_m) \subset \Sigma_\pi$ and $K(P_m^*) \subset \Sigma_\pi$, for $m \geq 2$.

Numerical experiments carried out for the polynomials $p \in \mathcal{S}$ of degree ≤ 7 (see Table 1 and [17, Table 1]) appear to confirm that these polynomials satisfy the conditions of Theorem 5.5, i.e., it appears for each polynomial p that the orbit of every critical point either diverges to infinity or is eventually in $1.1\overline{\mathbb{D}} \cup (-2, 2) \cup i(-2, 2)$. The same appears true for the polynomial $p \in \mathcal{S}$ of degree 18 in Lemma 5.1 for which $K(p) \not\subset \overline{U(p)}$. Thus it appears, from numerical evidence and Theorem 5.5, that $K(p) \subset \Sigma_\pi$ for these examples. These numerical experiments and Corollary 5.6 motivate a conjecture that $K(p) \subset \Sigma_\pi$ for all $p \in \mathcal{S}$.

6. Open Problems

We finish this paper with a note of open problems regarding the spectrum of the Feinberg-Zee random hopping matrix, particularly problems that the above discussions have highlighted. We recall first that [3] made several conjectures regarding Σ . It was proved in [16] that $\overline{\sigma_\infty} = \Sigma_\pi$, but the following conjectures remain open:

1. $\Sigma_\pi = \Sigma$;
2. Σ is the closure of its interior;
3. Σ is simply connected;
4. Σ has a fractal boundary.

Of these conjectures, perhaps the first has the larger implications. Certainly, if $\Sigma = \Sigma_\pi$, then we have noted below (3.2) that we have constructed already convergent sequences of upper (Σ_n^*) and lower (Π_n) bounds for Σ that can both be computed by calculating eigenvalues of $n \times n$ matrices. Further, if $\Sigma = \Sigma_\pi$, then the second of the above conjectures follows from Theorem 4.6.

The last three conjectures in the above list were prompted in large part by plots of π_n in [3], the plot of π_{30} reproduced in Figures 2 and 3. It is plausible that these plots, in view of (3.2), approximate Σ_π . We see no clear route to establishing the third conjecture above. Regarding the fourth, we note that the existence of the set \mathcal{S} of polynomial symmetries satisfying (3.3) suggests a self-similar structure to π_∞ and to Σ_π and Σ and their boundaries. Further, [17] has shown that Σ_π contains the Julia sets of all polynomials in \mathcal{S} , and Proposition 5.5 and Corollary 5.6 show that Σ_π contains the filled Julia sets, many of which have fractal boundaries, of the polynomials in an infinite subset of \mathcal{S} .

Regarding these polynomial symmetries we make two further conjectures:

5. $K(p) \subset \Sigma_\pi$ for all $p \in \mathcal{S}$;
6. $p^{-1}(\Sigma) \subset \Sigma$ for all $p \in \mathcal{S}$.

This last conjecture follows if $\Sigma = \Sigma_\pi$, by Theorem 3.2 from [17]. Further (see the discussion below (3.4)), it was shown in [5] that $p^{-1}(\Sigma) \subset \Sigma$ for the only polynomial of degree 2 in \mathcal{S} , $p(\lambda) = \lambda^2$.

The major subject of study and tool for argument in this paper has been Hagger's set of polynomial symmetries \mathcal{S} . We finish with one final open question raised immediately before Section 3.1.

7. Does \mathcal{S} capture all the polynomial symmetries of Σ ? Precisely, are there polynomial symmetries, satisfying (3.3), that are not either in \mathcal{S} or compositions of elements of \mathcal{S} ?

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